

The heat operator in infinite dimensions

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ABSTRACT. Let (H, B) be an abstract Wiener space and let μ_s be the Gaussian measure on B with variance s . Let Δ be the Laplacian (*not* the number operator), that is, a sum of squares of derivatives associated to an orthonormal basis of H . I will show that the heat operator $\exp(t\Delta/2)$ is a contraction operator from $L^2(B, \mu_s)$ to $L^2(B, \mu_{s-t})$, for all $t < s$. More generally, the heat operator is a contraction from $L^p(B, \mu_s)$ to $L^q(B, \mu_{s-t})$ for $t < s$, provided that p and q satisfy

$$\frac{p-1}{q-1} \leq \frac{s}{s-t}.$$

I give two proofs of this result, both very elementary.

1. INTRODUCTION

The heat operator, both on Euclidean space and on Riemannian manifolds, is a basic tool in finite-dimensional analysis. In infinite-dimensional analysis, the Laplacian (i.e., the most naive infinite-dimensional generalization of the finite-dimensional Laplacian) cannot be defined as a self-adjoint operator, because there is no such thing as Riemannian volume measure in infinite dimensions. If one replaces the nonexistent volume measure with a Gaussian measure or something similar on a nonlinear manifold, the Laplacian becomes not only fails to be self-adjoint but fails to be closable in L^2 . This makes it difficult to define the heat operator as a reasonable operator in L^2 .

In this paper, I will consider only the case of an infinite-dimensional Euclidean space. In that case, I present one possible way of making the heat operator into a well-defined, bounded operator.

I thank Professor Leonard Gross for valuable discussions, especially in pushing me to understand the relationship between the heat semigroup and the Hermite semigroup in terms of commutation relations.

2. GAUSSIAN MEASURES

Let H be an infinite-dimensional, real, separable Hilbert space. Since there does not exist anything like Lebesgue measure in infinite dimensions, we may consider instead a Gaussian measure. Let us attempt to construct, then, a “standard” (i.e., mean zero, variance one) Gaussian measure on H . This should be given by the nonrigorous expression

$$d\mu(x) = \frac{1}{Z} e^{-\|x\|^2/2} \mathcal{D}x, \tag{1}$$

where $\mathcal{D}x$ is the nonexistent Lebesgue measure on H and Z is a normalization constant. Unfortunately, this formal expression does not correspond to any well-defined measure on H . Specifically, (1) *can* be used to assign a “measure” to cylinder sets, but this set function does not have a countably additive extension to the generated σ -algebra. (A

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cylinder set is a set that can be described in terms of finitely many continuous linear functionals on H .)

To rectify this situation, we follow the approach of L. Gross in [Gr]. We introduce a Banach space B together with a continuous embedding of H into B with dense image. If B is “enough bigger” than H , in a sense spelled out precisely in [Gr], then there is a measure μ on B that captures the essence of the formal expression in (1). One rigorous way to characterize the measure μ is to use (1) to define a set function on cylinder sets *in the larger space B* . This set function, unlike the one on cylinder sets in H , has a countably additive extension to the generated σ -algebra. (This result, of course, assumes Gross’s condition on the embedding of H into B .) The original space H turns out to be a set of μ -measure 0 inside B . The book of Kuo [Ku] is an excellent standard reference on this material.

A pair (H, B) satisfying Gross’s condition is called an *abstract Wiener space*. The prototypical example is the one in which H is the space of H^1 functions on $[0, 1]$, equaling 0 at 0, with inner product given by

$$\langle f, g \rangle_H = \int_0^1 f'(x)g'(x) dx.$$

In this case, one may take B to be the space of continuous functions on $[0, 1]$ equaling 0 at 0. In this case, μ is the (concrete) Wiener measure, describing the behavior of Brownian motion.

As an alternative to using (1), one can characterize μ in terms of its Fourier transform. The measure μ is the unique one such that for all continuous linear functionals ϕ on B , we have

$$\int_B e^{i\phi(x)} d\mu(x) = e^{-\|\phi\|_H^2/2}, \quad (2)$$

where $\|\phi\|_H$ is the norm of ϕ as a linear functional on H (*not* B). (That is, $\|\phi\|_H$ is the norm of the restriction of ϕ to H .) In light of the standard formula for the Fourier transform of a Gaussian, (2) is formally equivalent to (1).

Note from either (1) or (2) that it is the geometry of H , rather than the geometry of B , that is controlling the Gaussian measure μ . One should think of μ as the standard Gaussian measure “on” H , where the larger space B is a technical necessity, needed to capture the measure.

We can also introduce Gaussian measures with variance s . For any $s > 0$, there is a unique measure μ_s on B such that

$$\int_B e^{i\phi(x)} d\mu_s(x) = e^{-\|\phi\|_H^2/2s}$$

for all continuous linear functionals ϕ on B . The measure μ_s is simply a dilation of μ .

3. THE LAPLACIAN AND THE HEAT OPERATOR

If we keep our focus on the geometry of H (rather than B), then we can introduce a Laplacian on B as follows. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for H , with the property that the linear functionals $x \rightarrow \langle e_n, x \rangle$ extend continuously from H to B . Then let $\{x_n\}_{n=1}^\infty$ be the coordinate functions (on B) associated to this basis; that is, $x_n = \langle e_n, x \rangle$ for $x \in B$. Then we define the Laplacian to be the operator given by

$$\Delta = \sum_{n=1}^\infty \frac{\partial^2}{\partial x_n^2}. \quad (3)$$

This operator is defined, for example, on polynomials, that is functions that can be expressed as polynomials in some finite collection of the x_n 's. Note that this operator is really the Laplacian and *not* the frequently considered number operator (also known as the Ornstein–Uhlenbeck or Dirichlet form operator). That is, (3) is the naive infinite-dimensional generalization of what is usually called the Laplacian on \mathbb{R}^d . (Those who are emotionally attached to the number operator should not lose heart; that operator will have its role to play later on.)

We would like to try to define the Laplacian as an unbounded operator in the Hilbert space $L^2(B, \mu)$ or more generally $L^2(B, \mu_s)$. In fact, Δ can be defined densely in $L^2(B, \mu_s)$, by, for example, defining it on polynomials. Unfortunately, though, the Laplacian defined in this way is a nonclosable operator. (This is equivalent to saying that the adjoint operator is not densely defined. Note that Δ is not self-adjoint in L^2 with respect to a Gaussian measure, even in finite dimensions.) Nonclosable operators are generally considered to be pathological; not only does such an operator fail to have a densely defined adjoint, but it is difficult to make any canonical choice of what its domain should be.

The nonclosability of the Laplacian can easily be seen by example. Define functions $f_n \in L^2(B, \mu_s)$ by

$$f_n(x) = \frac{1}{n} \sum_{k=1}^n (x_k^2 - s).$$

Since, as is easily verified, $\langle x_k^2 - s, x_l^2 - s \rangle_{L^2(B, \mu_s)} = 0$ for $k \neq l$, we can see that

$$\|f_n\|_{L^2(B, \mu_s)}^2 = \frac{c}{n}.$$

Thus, $f_n \rightarrow 0$ in $L^2(B, \mu_s)$ as n tends to infinity. On the other hand, a simple calculation shows that

$$\Delta f_n = 2 \quad (\text{constant function}).$$

Thus, the pair $(0, 2)$ is in the closure of the graph of Δ , which shows that Δ is not closable.

Since the Laplacian Δ is not closable, we cannot expect the heat operator $e^{t\Delta/2}$ to be any sort of reasonable semigroup in $L^2(B, \mu_s)$. One can define $e^{t\Delta/2}$ on polynomials as a (terminating) power series in Δ , but the resulting operator is again not closable. (This shows, in particular, that the heat operator is not bounded.) The same example functions f_n demonstrate the nonclosability of the heat operator. After all, since Δf_n is constant, $\Delta^2 f_n$ is zero and so $e^{t\Delta/2} f_n = f_n + t\Delta f_n/2$. Thus, $e^{t\Delta/2} f_n$ tends to t , whereas f_n tends to zero.

4. THE SEGAL–BARGMANN TRANSFORM

The discussion in the previous section shows that we cannot regard the heat operator $e^{t\Delta/2}$ as a reasonable operator from $L^2(B, \mu_s)$ to *itself*. If, then, we are going to make $e^{t\Delta/2}$ into a reasonable (preferably bounded) operator, then we must regard it as mapping from $L^2(B, \mu_s)$ to *some other space*. One way to do this is to look at the Segal–Bargmann transform. This transform (from one point of view) consists of applying the heat operator $e^{t\Delta/2}$ to a function and then analytically continuing the resulting function $e^{t\Delta/2} f$ in the space variable. Even in the infinite-dimensional case, this makes sense at least on polynomials. One can then prove an isometry formula that allows one to extend $e^{t\Delta/2}$ to a bounded operator from $L^2(B, \mu_s)$ into an appropriate Hilbert space of holomorphic functions on the complexification of B .

Theorem 1. *Let \mathcal{P} denote the space of polynomials inside $L^2(B, \mu_s)$. For all $t < 2s$, there is a Gaussian measure $\mu_{s,t}$ on $B_{\mathbb{C}} := B + iB$ such that the map*

$$f \rightarrow \text{analytic continuation of } e^{t\Delta/2} f,$$

as defined on polynomials, is isometric from $\mathcal{P} \subset L^2(B, \mu_s)$ into $L^2(B_{\mathbb{C}}, \mu_{s,t})$.

This is Theorem 4.3 of [DH]. Here, again, the analytic continuation is in the space variable (from B to $B_{\mathbb{C}}$) with t fixed. This theorem shows that $e^{t\Delta/2}$ extends continuously to an isometric map of $L^2(B, \mu_s)$ into $L^2(B_{\mathbb{C}}, \mu_{s,t})$. In [DH], it is shown that the image of this extended map is precisely the L^2 closure of the holomorphic polynomials in $L^2(B_{\mathbb{C}}, \mu_{s,t})$.

5. “TWO WRONGS MAKE A RIGHT”

In the preceding section, we saw that the heat operator $e^{t\Delta/2}$, followed by analytic continuation, can be regarded as a bounded (even isometric) map of $L^2(B, \mu_s)$ into a Hilbert space of “holomorphic” functions on $B_{\mathbb{C}}$, provided that $t < 2s$. (Here “holomorphic” means “belonging to the L^2 closure of holomorphic polynomials.”) We may ask, however, whether it is possible to regard the heat operator itself, without the analytic continuation, as a bounded operator from $L^2(B, \mu_s)$ into some space of functions on B . The answer, as we shall see in this section, is yes. The key is to regard the heat operator as mapping from $L^2(B, \mu_s)$ into a space defined using a Gaussian measure with a *different variance*. Specifically, we will see that for $t < s$, $e^{t\Delta/2}$ is a bounded operator from $L^2(B, \mu_s)$ into $L^2(B, \mu_{s-t})$.

Now, ordinarily, such a “changing of the variance” is not a good idea. That is, the identity map, which simply regards a function $f \in L^2(B, \mu_s)$ as an element of $L^2(B, \mu_{s-t})$, is highly ill defined. After all, it is known that the measures μ_s and μ_{s-t} ($0 < t < s$) are mutually singular; each measure is supported on a set that has measure zero with respect to the other measure. Thus, two functions that are equal μ_s -almost everywhere may not be equal μ_{s-t} -almost everywhere. Thus, the map $f \rightarrow f$ is not well defined from $L^2(B, \mu_s)$ to $L^2(B, \mu_{s-t})$, because elements of L^2 are not functions but rather equivalence classes of almost-everywhere equal functions. Alternatively, one can define the identity map on polynomials (mapping a polynomial in $L^2(B, \mu_s)$ to the same polynomial in $L^2(B, \mu_{s-t})$) and then check that this map is not closable (consider again the functions f_n).

We see, then, that the heat operator is not well defined from $L^2(B, \mu_s)$ to itself and that the identity map is not well defined from $L^2(B, \mu_s)$ to $L^2(B, \mu_{s-t})$. Nevertheless, when we put these two maps together, two wrongs turn out to make a right: the heat operator is a well-defined and bounded map from $L^2(B, \mu_s)$ to $L^2(B, \mu_{s-t})$. Somehow, for $f \in L^2(B, \mu_s)$, if we regard $e^{t\Delta/2} f$ as belonging to $L^2(B, \mu_{s-t})$ rather than $L^2(B, \mu_s)$, things work out better. With this point of view, $e^{t\Delta/2}$ actually becomes a bounded operator.

Actually, more than this can be said. The heat operator is actually bounded (even contractive) from $L^p(B, \mu_s)$ to $L^q(B, \mu_{s-t})$, for certain pairs (p, q) with $q > p$.

Theorem 2. *Fix $t < s$ and numbers $p, q > 1$ such that*

$$\frac{q-1}{p-1} \leq \frac{s}{s-t}.$$

Then the heat operator, initially defined on polynomials, extends to a contractive operator from $L^p(B, \mu_s)$ to $L^q(B, \mu_{s-t})$.

Note that since s is greater than $s - t$, the condition on p and q allows q to be greater than p . In particular, $p = q$ is always permitted. Theorem 2 is the main result of this paper. I will present two different proofs, in the two following sections.

6. PROOF USING HERMITE POLYNOMIALS

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ be an infinite multi-index, in which all but finitely many of the α_j 's are zero. A **polynomial** is then a finite linear combination of functions of the form $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots$. We let \mathcal{P} denote the space of all polynomials. Then inside $L^2(B, \mu_s)$ we define a **Hermite polynomial** to be a polynomial of the form

$$h_{\alpha,s}(x) := e^{-s\Delta/2}(x^\alpha). \quad (4)$$

Here, $e^{-s\Delta/2}$ is the *backward* heat operator, which is defined on any polynomial by a terminating power series in powers of Δ . The formula (4) is one of many equivalent ways of defining the Hermite polynomials. It is known that the Hermite polynomials form an orthogonal basis for $L^2(B, \mu_s)$, as α varies over all multi-indices of the above sort. The normalization is as follows:

$$\|h_{\alpha,s}\|_{L^2(B, \mu_s)}^2 = \alpha! s^{|\alpha|},$$

where $\alpha! = \alpha_1! \alpha_2! \dots$. Furthermore, for $1 < p < \infty$, the span of the Hermite polynomials is dense in $L^p(B, \mu_s)$.

We note that for any t and s we have

$$e^{t\Delta/2} e^{-s\Delta/2} = e^{-(s-t)\Delta/2},$$

by the usual power series argument. Thus for $s < t$, we have

$$e^{t\Delta/2}(h_{\alpha,s}) = h_{\alpha,s-t}.$$

This means that the time- t heat operator maps the Hermite polynomials that go with the Hilbert space $L^2(B, \mu_s)$ to the Hermite polynomials that go with the Hilbert space $L^2(B, \mu_{s-t})$. This simple observation provides the first indication that the “right” way to think of $e^{t\Delta/2}$ is as an operator from a function space defined using μ_s to a function space defined using μ_{s-t} .

Actually, $e^{t\Delta/2}$ maps an orthogonal basis for $L^2(B, \mu_s)$ to an orthogonal basis for $L^2(B, \mu_{s-t})$. Furthermore, since $(s-t)^{|\alpha|} \leq s^{|\alpha|}$, it follows easily that $e^{t\Delta/2}$ extends to a contractive mapping of $L^2(B, \mu_s)$ to $L^2(B, \mu_{s-t})$.

To establish the L^p to L^q properties in Theorem 2, we use scaling. We have said that the identity map (i.e., the map $f \rightarrow f$) is not well defined from $L^2(B, \mu_s)$ to $L^2(B, \mu_{s-t})$, or vice versa, because the measures μ_s and μ_{s-t} are mutually singular. There is, however, a nice map from $L^2(B, \mu_{s-t})$ to $L^2(B, \mu_s)$, or more generally of $L^p(B, \mu_{s-t})$ to $L^p(B, \mu_s)$, consisting of dilation. That is, if we define $D_{s,t} : L^2(B, \mu_{s-t}) \rightarrow L^2(B, \mu_s)$ by

$$(D_{s,t}f)(x) = f\left(\sqrt{\frac{s-t}{s}}x\right),$$

then this map is well defined and isometric from $L^p(B, \mu_{s-t})$ to $L^p(B, \mu_s)$, for all $1 \leq p \leq \infty$. This amounts to saying that μ_s can be obtained from μ_{s-t} by a dilation of B .

Meanwhile, how do Hermite polynomials transform under this dilation? Well, using the formula (4) for the functions $h_{\alpha,s}$, it is not hard to see that

$$D_{s,t}(h_{\alpha,s-t}) = \left(\frac{s-t}{s}\right)^{|\alpha|/2} h_{\alpha,s}. \quad (5)$$

Now, it makes sense to start with a Hermite polynomial $h_{\alpha,s} \in L^p(B, \mu_s)$, apply the heat operator to get $h_{\alpha,s-t} \in L^p(B, \mu_{s-t})$, and then apply the dilation $D_{s,t}$ to get back to $L^p(B, \mu_s)$. We have, by (5),

$$D_{s,t}e^{t\Delta/2}h_{\alpha,s} = D_{s,t}h_{\alpha,s-t} = \left(\frac{s-t}{s}\right)^{|\alpha|/2} h_{\alpha,s}. \quad (6)$$

Let us introduce the “number operator” N_s defined on polynomials by the condition

$$N_s h_{\alpha,s} = |\alpha| h_{\alpha,s}. \quad (7)$$

Then (6) can be rewritten as

$$\begin{aligned} D_{s,t} \circ e^{t\Delta/2} h_{\alpha,s} &= \left(\frac{s-t}{s}\right)^{N_s/2} h_{\alpha,s} \\ &= e^{-\tau N_s} h_{\alpha,s}, \end{aligned} \quad (8)$$

where

$$\tau = \frac{1}{2} \log \left(\frac{s}{s-t} \right). \quad (9)$$

Here $e^{-\tau N_s}$ is defined on polynomials by setting $e^{-\tau N_s} h_{\alpha,s} = e^{-\tau|\alpha|} h_{\alpha,s}$, in accordance with (7).

Now, every polynomial is a finite linear combination of Hermite polynomials. What (8) says, then, is that on the dense subspace \mathcal{P} of $L^2(B, \mu_s)$ we have

$$D_{s,t} \circ e^{t\Delta/2} = e^{-\tau N_s}. \quad (10)$$

Note that $e^{-\tau N_s}$ is a bounded operator on $\mathcal{P} \subset L^2(B, \mu_s)$, since its action on the orthogonal basis $\{h_\alpha\}$ consists of multiplying by $e^{-\tau|\alpha|}$. Thus (10) tells us that $D_{s,t} \circ e^{t\Delta/2}$ has an extension which is a bounded operator from $L^2(B, \mu_s)$ to itself.

We can, however, say more than this. Nelson [Ne] has shown that $e^{-\tau N_s}$ is contractive from $L^p(B, \mu_s)$ to $L^q(B, \mu_s)$, provided that

$$\tau \geq \frac{1}{2} \log \frac{q-1}{p-1}. \quad (11)$$

More precisely, this can be interpreted as saying that for τ satisfying the above condition, $e^{-\tau N}$ has extension from the space \mathcal{P} of polynomials to $L^p(B, \mu_s)$ that maps contractively into $L^q(B, \mu_s)$. (This contractivity property of $e^{-\tau N_s}$, where $q > p$ is permitted, is referred to as hypercontractivity.) The condition (11) is, in light of (9), equivalent to

$$\frac{s}{s-t} \geq \frac{q-1}{p-1}. \quad (12)$$

Thus, whenever (12) holds, $D_{s,t} \circ e^{t\Delta/2}$ has an extension that is a contractive map of $L^p(B, \mu_s)$ to $L^q(B, \mu_{s-t})$. Since $D_{s,t}$ is an isometric isomorphism of $L^q(B, \mu_{s-t})$ onto $L^q(B, \mu_s)$, this amounts to saying that $e^{t\Delta/2}$ has an extension that is a contractive map of $L^p(B, \mu_s)$ to $L^q(B, \mu_{s-t})$. This establishes Theorem 2.

7. THE HERMITE SEMIGROUP AND THE HEAT SEMIGROUP

Looking back on the argument in the preceding section, we see that the role of the Hermite polynomials is not essential. Rather, we used the Hermite polynomials to obtain the identity (10), at which point Theorem 2 is seen to be a consequence of Nelson's hypercontractivity theorem. That is, our result really hinges on a relationship between the heat semigroup (the operators $e^{t\Delta/2}$) and the Hermite semigroup (the operators $e^{-\tau N_s}$). What (10) is saying is that (at least on polynomials) the Hermite semigroup at time τ is the same as the heat semigroup at t , modulo a dilation, where t and τ are related as in (9). What we want to do in this section is explore two other ways (besides the Hermite polynomial argument of the previous section) of understanding this relationship between the two semigroups.

We begin by looking at the integral kernels for the two semigroups. In the finite-dimensional case, the heat semigroup $e^{t\Delta/2}$ can be computed as integration against the heat kernel, which is a Gaussian. Meanwhile, in the finite-dimensional case, the Hermite semigroup $e^{-\tau N_s}$ can be computed using the Mehler kernel, which is also a Gaussian, though of a slightly more complicated variety. See, for example, the article [Sj, Thm. 1] of Sjögren, which derives the formula for the Mehler kernel in a way that emphasizes the relationship with the heat kernel. From these formulas for the kernels, one can easily read off the identity (10) in any finite number of variables. This is sufficient to establish (10) on polynomials, since each polynomial is a function of only finitely many variables.

As an alternative to using the Hermite polynomial argument of the previous section or the argument in this section based on formulas for the integral kernels, we can explore the relationship between the heat semigroup and the Hermite semigroup using commutation relations. Although we have defined the number operator by its action on Hermite polynomials ($N_s h_{\alpha,s} = |\alpha| h_{\alpha,s}$), N_s can also be expressed as a differential operator, as follows. Let

$$D = \sum_{k=1}^{\infty} x_k \frac{\partial}{\partial x_k}.$$

Then $Dx^\alpha = |\alpha| x^\alpha$. From this it follows, in light of (4), that

$$e^{-s\Delta/2} D e^{s\Delta/2} h_{\alpha,s} = |\alpha| h_{\alpha,s}.$$

We now use the standard identity $e^A B e^{-A} = e^{\text{ad}_A}(B)$, where $\text{ad}_A(B) = [A, B]$. A simple calculation shows that

$$[\Delta, D] = 2\Delta. \tag{13}$$

Thus if we set

$$\begin{aligned} N_s &= e^{-s\Delta/2} (D) e^{s\Delta/2} \\ &= D - s\Delta, \end{aligned} \tag{14}$$

where $(\text{ad}_\Delta)^n D = 0$ for $n \geq 2$, N_s defined in this way will have the correct behavior on Hermite polynomials.

Our task, then, is to compute

$$e^{-\tau N_s} = \exp \{ \tau (s\Delta - D) \},$$

with the aid of the commutation relation (13). We will apply a special case of the Baker–Campbell–Hausdorff formula. Suppose that A and B are linear operators on a finite-

dimensional vector space and that $[A, B] = \alpha A$. Then we have

$$e^{\tau(A+B)} = e^{\tau B} \exp \left\{ \frac{e^{\tau\alpha} - 1}{\alpha} A \right\}. \quad (15)$$

To prove (15), let $Z(\tau)$ denote the quantity on the right-hand side. Using the identity

$$e^{\tau B} A e^{-\tau B} = e^{\tau \text{ad}_B}(A) = e^{-\tau\alpha} A,$$

it is not hard to show that Z satisfies the differential equation $dZ/d\tau = (A + B)Z(\tau)$. Since the left-hand side of (15) clearly satisfies the same differential equation and since the two sides are equal when $\tau = 0$, we conclude that the two sides are equal for all τ . (See, for example, Section 4 of [Di], where a slight variant of (15) is analyzed in the context of unbounded operators.)

We wish to apply (15) with $A = s\Delta$ and $B = -D$, in which case we would have $\alpha = -2$. Of course, we should not blindly apply results for operators on finite-dimensional spaces to operators on infinite-dimensional spaces. Fortunately, however, there is no problem in this instance, since any polynomial $p \in \mathcal{P}$ is contained in a finite-dimensional space that is invariant under both D and Δ and hence under N_s . (A polynomial p , by definition, involves only finitely many monomials x^α . Thus there is some n such that p is contained in the space of polynomials of degree at most n in some finite collection of variables x_1, \dots, x_m .)

On each finite-dimensional invariant subspace, then, we have

$$e^{-\tau N_s} = e^{-\tau D} \exp \left\{ \frac{e^{-2\tau} - 1}{(-2)} (s\Delta) \right\},$$

where we may compute using (9) that

$$-\frac{e^{-2\tau} - 1}{2} = -\frac{\frac{s-t}{s} - 1}{2} = \frac{t}{2s}.$$

Thus we get

$$e^{-\tau N_s} = e^{-\tau D} e^{t\Delta/2}. \quad (16)$$

It remains only to understand the factor of $e^{-\tau D}$ on the right-hand side of (16). Recall that $Dx^\alpha = |\alpha| x^\alpha$. Thus,

$$e^{-\tau D} x^\alpha = e^{-\tau|\alpha|} x^\alpha = (e^{-\tau} x)^\alpha, \quad (17)$$

where $e^{-\tau} = \sqrt{(s-t)/s}$. From this we can see that $e^{-\tau D} = D_{s,t}$ on polynomials and thus (16) is equivalent (on polynomials) to the identity (10).

8. CONCLUDING REMARKS

The boundedness properties of the heat operator given in Theorem 2 can be deduced from the relationship (10) between the heat semigroup and the Hermite semigroup. That relationship, in turn, can be understood in various ways, using Hermite polynomials, using the integral kernels, or using commutation relations. In the last approach, the identity (10) follows from the commutation relation (13) together with the special form (15) of the Baker–Campbell–Hausdorff formula.

We have seen in (17) that D is the generator of dilations. Thus the commutation relation between D and Δ in (13) reflects that the Laplacian transforms in a simple way

under dilations. This in turn reflects that the metric on Euclidean space transforms in a simple way under dilations.

What prospect, then, is there for proving some analog of Theorem 2 in some other setting, that is, on some (possibly infinite-dimensional) manifold other than Euclidean space? One possibility is to consider manifolds where there is some natural sort of dilation operators. For example, on the Heisenberg group there is a sub-Laplacian that behaves in a nice way with respect to certain nonisotropic “dilations.” Thus the Heisenberg group, whether in its finite- or infinite-dimensional form, is a natural candidate for proving a theorem similar to Theorem 2.

On the other hand, even if an infinite-dimensional manifold has no natural dilations, it is still conceivable that something like Theorem 2 might hold. Specifically, suppose there exists on an infinite-dimensional manifold M some natural sort of heat kernel measure μ_s , based at a fixed point in M . (One may think, for example, of path or loop groups and the heat kernel measures considered by Malliavin [Ma] and Driver–Lohrenz [DL], or various infinite-dimensional limits of finite-dimensional groups and the heat kernel measures of Gordina [Go1, Go2].) One might hope that in some cases, the heat operator $e^{t\Delta/2}$ could be a bounded operator from $L^p(M, \mu_s)$ to $L^q(M, \mu_{s-t})$, for $t < s$ and appropriate pairs (p, q) . One might begin by studying only the Hilbert space case and try to see whether $e^{t\Delta/2}$ is bounded (or contractive) from $L^2(M, \mu_s)$ to $L^2(M, \mu_{s-t})$. Of course, the methods of proof used in the present paper would not carry over to such a setting. Nevertheless, the proposed result is of a simple enough form that some other method of proof may be found.

The conclusion I would like to draw from all of this is that one should not give up on studying the heat operator associated to the true Laplacian, even in the infinite-dimensional setting. Here by “true” Laplacian I mean something like the Laplace–Beltrami operator associated to some Riemannian metric on an infinite-dimensional manifold, that is the $\nabla^*\nabla$ operator associated to the (fictitious) Riemannian volume measure. This is to be contrasted with something like a number operator or Ornstein–Uhlenbeck operator, which is the $\nabla^*\nabla$ operator associated to a Gaussian or heat kernel measure. Even though the Laplacian is bound to be a pathological operator, this should not cause us to give up on defining the heat operator. One merely has to look for the right interpretation of the heat operator, an interpretation that will allow it to make sense. One candidate for such an interpretation is to view the time- t heat operator as an operator from $L^2(M, \mu_s)$ to $L^2(M, \mu_{s-t})$.

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